Stochastic Models of the Errors in Orbital Predictions for Artificial Earth Satellites

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NATURAL satellites move in an environment that is essentially drag-free. Their motions are well described by the standard methods of celestial mechanics (1).[2] In particular, the errors in orbital predictions for natural satellites are well handled by the least-square fitting procedure developed by Gauss and Legendre, because the dominant source of error is observational. Artificial earth satellites, on the other hand, are subjected to a large and fluctuating air drag (2). For many artificial satellites it is this fluctuation in drag, rather than observational error, which is the main source of error in the one- to two-week orbital predictions (3).

It is the purpose of this note to calculate the errors in orbital predictions for cases in which drag fluctuations cannot be ignored. To facilitate the statistical analysis, the drag fluctuations are separated into a sinusoidal component, with a period of 27 days, and a random component for which two bounding correlation functions are used. In order to show the effect of drag fluctuations without the complications introduced by smoothing, a simplified model in which the initial orbit is perfectly known is first presented. Then a more realistic model, which includes the effects of observational errors and smoothing, is given. Errors in predictions calculated from this latter model are then compared with actual errors in orbital predictions.

Errors in Orbital Predictions Assuming Perfect Initial Elements

Orbital predictions are usually made by smoothing the observations over a number of revolutions to determine the

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orbital elements and the rate of change of period. Then these quantities are projected ahead for N revolutions to predict the time of the Nth equatorial crossing. The rate of change of period, D, is usually assumed to be a constant in any one prediction. If all perturbations except this constant drag are ignored, the satellite will return to its original position on the celestial sphere for the Nth time after a time interval $\Delta T_0 \simeq NP_0 + (N^2/2)\,D$, where P_0 is the initial period. However, the rate of change of period fluctuates about the smoothed value D, and these fluctuations cause ΔT_0 to be in error. The problem is to calculate the root-mean-square error in ΔT_0.

For simplicity, first ignore the uncertainties introduced by observational errors and smoothing, and suppose that the orbit is perfectly known at the initial time. The objective is to determine the effect of fluctuations in acceleration on knowledge of the future times of equatorial crossing. An examination of published graphs of satellite acceleration (2, 4, 5) suggests that the acceleration be separated into sinusoidal and random components for analytical purposes, because correlated and uncorrelated phenomena are not amenable to the same statistical treatment. Accordingly, the deviations from the smoothed rate of change of period D are separated into a sinusoidal component with a 27-day period and a random component ρ_n.

The distribution function and correlation function of the random fluctuations are unknown. (They probably depend on the height of perigee, eccentricity, local time of perigee, and condition of the sun.) Therefore, the error in the predicted time of equatorial crossing caused by random fluctuations is calculated for two different cases, which are believed to represent physical bounds on the problem. In the upper bound, the random drag fluctuations are assumed independent from one revolution to the next. In the lower bound, the random fluctuations are assumed perfectly correlated over intervals of 25 revolutions but uncorrelated from interval to interval. The calculation of the errors in orbital prediction caused by the random fluctuations in these two cases is performed in the Appendix. The results show that a simple approximation to the rms error in the predicted time of equatorial crossing (in minutes), N revolutions after the orbit was perfectly known, is given by

$$G_{\mathrm{rms}}(N) = 5\sigma(N^3/3)^{1/2} \qquad [1]$$

where the standard deviation of the random fluctuation σ (in minutes per revolution), calculated from observations smoothed over intervals of 25 revolutions, is given by the empirical relation $\sigma = 1.2 \times 10^{-3}\, h_p|D|$, where h_p is the height of perigee in nautical miles and D is the smoothed rate of change of period in minutes per revolution. (The expression for σ was derived from fluctuations in the accelerations of satellites having perigee heights between 120 and 350 naut miles.) Eq. [1], for the rms error due to random fluctuations, is between the two bounds derived in the Appendix and is asymptotic to both for large N.

The contribution of the sinusoidal drag variation to the error in the predicted time of equatorial crossing is derived in Ref. 6. The result is

$$H_{\mathrm{rms}}(N) = (2)^{-1/2}A(k)^{-2}\{[1 - \cos(kN) - (kN)^2/2]^2 + [kN - \sin(kN)]^2\}^{1/2} \qquad [2]$$

where H_{rms} is the rms sinusoidal prediction error (in minutes) for arbitrary initial phase of the sinusoidal drag, and $k = 2\pi\,(1/27)\,P/1440$, where P is the period in minutes. A is defined by the empirical relation $A = h_p|D| \times 10^{-3}$.

The sinusoidal and random errors can be combined to give $\Delta\tau$, the rms error in timing of an orbital prediction when the initial elements are perfect:

$$\Delta\tau(N) = [G_{\mathrm{rms}}^2(N) + H_{\mathrm{rms}}^2(N)]^{1/2} \qquad [3]$$

Errors in Orbital Predictions When the Elements and Rate of Change of Period are Obtained by Smoothing Observations

In the preceding simplified formulas, a perfect knowledge of the orbit at the initial time, or epoch, has been assumed. In actual orbital predictions, the elements at the epoch and the rate of change of period are usually found by some smoothing procedure, using data containing observational errors. In order to make the present problem tractable, the observations are taken to be uniformly distributed throughout the smoothing interval. Let there be M independent observations in a smoothing interval of i revolutions. Assume that there are three independent causes of errors in calculating the period and rate of change of period: a 27-day sinusoidal variation in the rate of change of period, a random fluctuation in the rate of change of period which is independent from revolution to revolution, and a measurement error introduced by the tracking device. The errors will be given as a function of the number of revolutions N after the epoch, which is taken to be at the *center* of the smoothing interval. The following results are derived in Ref. 6.

The contribution of the smoothed sinusoidal drag variation to the rms error in an orbital prediction that runs for N revolutions from the epoch is

$$S(N) = Ak^{-2}\,(2)^{-1/2}\,(\alpha^2 + \beta^2)^{1/2} \qquad [4]$$

where

$$\alpha = \cos kN - \left(\frac{2}{ik}\right)\sin\left(\frac{ik}{2}\right) + \frac{64}{i^3k}\sin\left(\frac{ki}{4}\right) \times \left[1 - \cos\left(\frac{ki}{4}\right)\right]\left[N^2 - i\,\frac{(i+2)}{12}\right]$$

and

$$\beta = \sin kN - kN + 8N[i(i+2)k]^{-1} \times \left[\cos\left(\frac{ki}{2}\right) - 1 + \frac{i^2k^2}{8}\right]$$

As the smoothing interval i approaches zero, Eq. [4] approaches Eq. [2] for the sinusoidal error when there is no smoothing.

The contribution of the smoothed random fluctuations to the rms error in orbital prediction is

$$R(N) = 5\sigma\{(N^3/3) + 2(i/4)^3[\tfrac{64}{5}(N/i)^4 - 16(N/i)^3 + (N/i)^2]\}^{1/2} \qquad [5]$$

for $N \geqslant i/2 \gg 1$. This equation should be compared with its unsmoothed counterpart, Eq. [1].

The contribution of smoothed measurement errors to the rms error in the predicted time of the Nth equatorial crossing is

$$0(N) = \sigma_0(M)^{-1/2}(i)^{-2}\{(i)^4[M(M+2)^{-1} + (16/9)(M+2)^2/M^2] + 256\,N^4 + 32\,Ni[(i)^2/(3M) - 4\,N^2\,(M+2)^{-1}] + 16\,(Ni)^2[M(M+2)^{-1} - (8/3)(M+2)/M - 2M\,(M+2)^{-2}]\}^{1/2} \qquad [6]$$

where all the observations are assumed to have the same standard deviation σ_0 expressed as an equivalent timing error in minutes.

Assuming that the observational, sinusoidal, and random errors are independent, they can be combined to give

$$E_{\mathrm{rms}}(N) = \{[O(N)]^2 + [S(N)]^2 + [R(N)]^2\}^{1/2} \qquad [7]$$

where $E_{\mathrm{rms}}(N)$ is the standard deviation (in minutes) of the predicted time of the Nth equatorial crossing after the epoch, when the elements and rate of change of period are obtained by smoothing observations. $E_{\mathrm{rms}}(N)$ represents the error tangential to the orbit of the satellite projected on the celestial sphere. Errors at right angles to the orbit are usually an order of magnitude smaller.

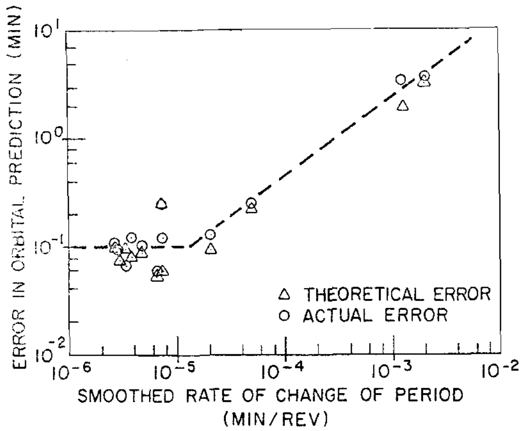

Fig. 1 Errors in one- to two-week predictions vs smoothed rate of change of period

Errors in actual one- to two-week predictions issued by the Vanguard Computing Center and NASA Computing Center are compared with the theoretical model, represented by Eq. [7], in Fig. 1. Many of these "experimental" values are tabulated in Ref. 3. The sloping line in Fig. 1 fits the errors of prediction in the regime in which drag fluctuations are the dominant cause of error. The horizontal line fits where observational errors are dominant. The horizontal line reflects the accuracy and abundance of Minitrack observations. Predictions based on a different type of observation would have a different horizontal line.

Appendix: Contribution of Random Drag Fluctuations to Error in Predicted Time of Equatorial Crossing, Assuming Perfect Initial Elements

Consider that the times of equatorial crossing are predicted by assuming a constant rate of change of period. But suppose that there are random fluctuations about the average change in period. Let these random fluctuations be $\rho_1, \rho_2, \ldots, \rho_i, \ldots, \rho_N$. Then after N revolutions, the error in the predicted time will be

$$E(N) = -\sum_{n=1}^{N}\sum_{j=1}^{n}\rho_j = -[N\rho_1 + (N-1)\rho_2 + \ldots + \rho_N] \qquad \text{[A1]}$$

If each ρ_i is independent and has the standard deviation F, then the standard deviation of $E(N)$ is

$$E(N)_{\text{rms}} = F\left[\sum_{n=1}^{N} n^2\right]^{1/2} = F\left[N(N+1)\,\frac{2N+1}{6}\right]^{1/2} \qquad \text{[A2]}$$

On the other hand, suppose that the random drag fluctuations are perfectly correlated over intervals of λ revolutions but independent from one interval to the next. (λ will later be set equal to 25, because that is the usual smoothing interval in published orbits.) Since the accelerations are assumed to be correlated over intervals of λ revolutions, $\rho_1 = \rho_2 = \ldots = \rho_q = \rho_A$, $\rho_{q+1} = \rho_{q+2} = \ldots = \rho_{q+\lambda} = \rho_B$, $\rho_{q+\lambda+1} = \rho_{q+\lambda+2} = \ldots = \rho_{q+2\lambda} = \rho_C$, etc. The possible values of q range from 1 to λ. In the absence of particular information, all values of q will be assigned equal weights. When $j = 1$, $\rho_j = \rho_A$. When $j = 2$, ρ_j will equal ρ_A if $2 \le q \le \lambda$, and $\rho_j = \rho_B$ if $q = 1$. When $j = 3$, $\rho_j = \rho_A$ if $3 \le q \le \lambda$, and $\rho_j = \rho_B$ if $q = 1$ or 2, etc. The equal weighting of the λ values of q can be expressed by averaging over the ensemble of possible values: $\rho_1 = \rho_A$, $\rho_2 = (1/\lambda)[(\lambda - 1)\rho_A + \rho_B]$, $\rho_3 = (1/\lambda)[(\lambda - 2)\rho_A + 2\rho_B]$, etc. The timing error, averaged over the ensemble of possible values of q, will be called $\overline{E(N)}$. It is found by substituting these ρ_j's into Eq. [A1]. The resulting expressions for the rms error in prediction in the correlated case are

$$[\overline{E(N)}]_{\text{rms}} = \{[N(N+1)\sigma]/(6\lambda)\} \times [9\lambda^2 - 6\lambda(N-1) + 2(N-1)^2]^{1/2} \quad N \le \lambda \qquad \text{[A3]}$$

and

$$[\overline{E(N)}]_{\text{rms}} = (\sigma/6\lambda)\,\{Q + \lambda^2(\lambda+1)^2 \times (3N - \lambda + 1)^2 + \xi^2(\xi-1)^2(\xi+1)^2 + [2\lambda(\lambda^2 - 1) + \nu(1 + 6\lambda\xi - \nu^2)]^2\}^{1/2} \quad N \ge \lambda \qquad \text{[A4]}$$

where $\xi = N - \mu\lambda$, $\nu = \xi + \lambda$, and

$$Q = 0 \qquad N < 2\lambda$$

$$= 36\lambda^4 \sum_{K=1}^{\mu-1}(N - K\lambda)^2 \qquad N \ge 2\lambda$$

and μ is the largest integer for which $N - \mu\lambda \ge 0$.

Limits of the equations for correlated and uncorrelated fluctuations

One would expect the correlated and uncorrelated models to approach the same limiting form when predictions run for a time interval that is long compared with the correlation time of the correlated model. The coefficient F of the random fluctuations (which is not directly observable) could then be calculated in terms of the coefficient σ of the correlated fluctuations (which is observable). If one takes the limit of Eq. [A2] as N approaches ∞, one obtains

$$F[N^3/3]^{1/2}$$

The limit of Eq. [A4] is

$$(\lambda)^{1/2}\,\sigma(N^3/3)^{1/2}$$

Thus, the limits for uncorrelated and correlated errors have the same asymptotic form. Now set $\lambda = 25$, because satellite accelerations are usually smoothed over 25 revolutions. This makes it possible to evaluate the constant F, which must equal 5σ. The relationship $F = 5\sigma$ corresponds exactly to the situation in the theory of errors, in which the standard deviation of the mean of λ independent observations equals the standard deviation of one observation divided by the square root of λ. The detailed shape of the correlation function is not very important for predictions whose duration exceeds the correlation time. The asymptotic form is a convenient approximation to the error contributed by random fluctuations, when the initial elements are perfect.

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